Similarity in swirling wakes and jets

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Both the net linear momentum and the net angular momentum of a developing swirling flow can play important parts in determining its ultimate form. To illustrate this the turbulent wake with both axial and swirl components of mean velocity is discussed, in particular for the two limiting cases of domination by linear momentum and domination by angular momentum. The dual conservation of axial and angular momentum implies that in general the mean swirl component decreases more rapidly downstream than does the defect in the mean axial velocity. Hence wakes with non-zero momentum flux ultimately have the familiar length scale $\sim z^{\frac{1}{3}}$ and velocity defect scale $\sim z^{-\frac{2}{3}}$. But in the wake of a self-propelled body the net drag is negligible and a swirl-dominated development can persist with length scale $\sim z^{\frac{1}{4}}$ and swirl velocity scale $\sim z^{-\frac{3}{4}}$.

There are a number of nearly rectilinear flows, laminar and turbulent jets and wakes, whose asymptotic forms are dependent on only a few parameters: for a wake the net drag and uniform convective velocity, for a jet the net thrust, and for laminar flows the fluid viscosity. Once account is taken of the way in which the controlling parameters are grouped in the simplified equations governing each situation, only one length and one velocity scale can be formed using the prescribing quantities. These scales provide a skeleton of the development of each flow.

The asymptotic forms for swirling wakes and jets are determined not only by their net linear momentum but also by their net angular momentum. With this parameter added, it is possible to define two scale lengths and two scale velocities. No longer is there a clear prescription of the variation of width and intensity. Instead, the similarity condition takes the form of an unprescribed functional relationship between two non-dimensional parameters, one characterizing the linear momentum, the other the angular momentum. The form of the non-dimensional groups is determined by the arrangement of the controlling parameters in the appropriate momentum equations and integrals, as well as by the dimensional requirement.

But in the two limiting cases one of the parameters drops out and more explicit results can be found. These are the well-known asymptotic scales for wakes and jets controlled by the linear momentum flux, and another set of scales for flows determined by the angular momentum flux.

Here this approach is exemplified by the study of the axisymmetric turbulent wake; in this case the swirl-dominated limit has an important application to the wakes of self-propelled bodies. There are several other flows that can be analysed in a broadly similar manner. The most obvious is the laminar wake with swirl. Laminar and turbulent axisymmetric jets may also be considered. However, the swirl-dominated limit is apparently not consistent with a nearly cylindrical jet concentrated near an axis, but finds its place more naturally in the study of a radial jet concentrated near a plane. Radial wakes formed in source flow past a toroidal obstacle also exhibit the two limiting kinds of domination.

The turbulent swirling wake

Well away from the generating body a turbulent swirling wake in an incompressible fluid is quite accurately described by the simplified equations:

$$W_1\frac{\partial W}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\overline{uw}) = 0 \quad \text{for axial momentum,}$$

$$W_1\frac{\partial V}{\partial z}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{uv}) = 0 \quad \text{for tangential momentum,}$$

where W_1 is a uniform convective velocity, V and W are the mean values of swirl and axial velocity components, $-\overline{uv}$ and $-\overline{uw}$ are the kinematic mixing stresses. The flow is further specified by the two invariants:

$$W_1\int_0^\infty (W_1-W)\,r\,dr = F \quad \text{for axial momentum,}$$

$$W_1\int_0^\infty Vr^2dr = G \quad \text{for angular momentum.}$$

We consider solutions that are determined solely by the parameters F, G and W_1. Such solutions may be expected to describe the motion in any wake far from the generating body. For such flows the wake width is given by

$$l = l(F, G, W_1, z).$$

But these parameters occur in the specifying equations only in the combinations F/W_1, G/W_1, z/W_1. Hence, on dimensional grounds,

$$f\left(\frac{Fz}{W_1^2l^3}, \frac{Gz}{W_1^2l^4}\right) = 0,$$

$$\frac{Fz}{W_1^2l^3} = g\left(\frac{Fl}{G}\right) \quad \text{or} \quad \frac{Gz}{W_1^2l^4} = h\left(\frac{Fl}{G}\right).$$

Two limiting cases may be considered.

A. Control by linear momentum; $Fl/G \gg 1$. Here we may expect the development to be independent of G. Then

$$l \sim (Fz/W_1^2)^{\frac{1}{3}} \sim z^{\frac{1}{3}},$$

and the scale of the velocity defect is found to be $w \sim z^{-\frac{2}{3}}$, using the axial momentum invariant or another similarity analysis. These are the well-known results for a turbulent axisymmetric wake.

B. Control by angular momentum; $Fl/G \ll 1$. Now F may be expected to play no part. Then

$$l \sim (Gz/W_1^2)^{\frac{1}{4}} \sim z^{\frac{1}{4}},$$

with swirl velocity scale $v \sim z^{-\frac{3}{4}}$ to keep the net angular momentum constant. The virtual origins for the cases A and B will in general be different.

Note the linking of F and l in the parameter Fl/G. As a wake spreads it becomes more and more like one whose linear momentum is very large. Further, using the invariants we can make the estimate

$$w/v \sim Fl/G,\dagger$$

which suggests that as a wake spreads the mean swirl component will always decrease more rapidly than the mean axial velocity defect. Ultimately any swirling wake in which $F \not\equiv 0$ will develop following the scales A, although its early development may be of the swirl-dominated form B. But in the wake of a self-propelled body $F \equiv 0$; the mode B may then persist throughout.

It can be shown that both kinds of development are consistent with the simplifications made in the momentum equations and integrals; the neglected terms decrease more rapidly downstream than do those retained. In both kinds of wake the Reynolds number characterizing the flow decreases downstream. In neither can turbulence be maintained indefinitely.

We may note that the two important production terms of the equation for the turbulence energy are

$$-\overline{uw}\frac{\partial W}{\partial r} \quad \text{and} \quad -\overline{uv}r\frac{\partial}{\partial r}\left(\frac{V}{r}\right).$$

The first is of prime importance in the development A, while the second provides most of the turbulence energy in case B. In this way the turbulence is linked with the mean velocity distribution; in the two limiting cases it develops with the scales given above for the mean velocities. It was in fact a consideration of the turbulent energy balance that made apparent the possibility of two simple asymptotic modes. Dr Ian Proudman pointed out that less specific arguments could be applied.

Similarity arguments give no further definite information about the variation of the scales when $Fl/G \sim O(1)$. However, a simple interpolation such as

$$l = (G/F)\,(\alpha p^{\gamma/3} + \beta p^{\gamma/4})^{1/\gamma}, \quad \text{where} \quad p = F^4 z/G^3 W_1^2$$

with α, β, and γ constants, may prove useful in practice.

† An estimate only, since the non-dominant component of the mean velocity need not have a self-preserving distribution even though the dominant component does.